Indian Statistical Institute, Bangalore

B.Math (Hons.) III Year / M.Math II Year, Second Semester

Semestral Examination

Combinatorics and Graph Theory

Time: 3 hours | May 04, 2011 | Instructor: B.Bagchi

Maximum marks: 100

1. Decide whether there is a strongly regular graph with parameters (i) $(9, 4, 2, 1)$ and (ii) $(9, 4, 1, 2)$. [5+5 =10]

2. Let G be a finite graph of diameter d and girth $2d + 1$.

   a) Let x, y be two vertices of G at distance d and let P be a path of length d joining x and y. For any neighbour x' of x which is not on P, show that there is a unique neighbour y' of y such that y' is not on P and such that x' and y' are at distance d.

   b) If C is a $(2d+1)-$ cycle in G then use part (a) to prove that all the vertices of C have the same degree in G. [10+10 =20]

3. Let q be a prime power.

   a) Sketch a proof of the existence and uniqueness of the field $\mathbb{F}_q$ of order q.

   b) If $q \equiv 3 \pmod 4$ and $\mathbb{F}_q^0$ is the subgroup of index 2 in $\mathbb{F}_q^*$, then show that $x + \mathbb{F}_q^0, x\varepsilon\mathbb{F}_q$, are the blocks of a 2- design, and compute its parameters. [10+10 =20]

4. a) Let B be a collection of k- subsets of a v- set such that any two distinct members of B have at most $t - 1$ elements in common $(0 < t < k < v)$. Then show that $\sharp(B) \leq \binom{v}{t}/\binom{k}{t}$.

   b) If equality holds in (a) then show that $\binom{k-s}{t-s}$divides $\binom{v-s}{t-s}$ for all $s, 0 \leq s < t$. [8+12 = 20]

5. a) For any positive integer n, let Ω_n be the set of real matrices of order $n \times n$ all whose entries are in the interval $[-1, 1]$. Let $a_n =$ max $\{det(A); A\varepsilon\Omega_n\}$. Then show that there is an $n \times n$ matrix H such that all entries of H are ±1 and det $(H) = a_n$.

   b) An $n \times n$ complex matrix H is called a complex Hadamard matrix if all the entries of H have modulus $= 1$ and $HH^* = nI = H^*H$. Construct a 3×3 complex Hadamard matrix. [20+10=30]